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On residuated skew lattices

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Abstract

In this paper, we define residuated skew lattice as non-commutative generalization of residuated lattice and investigate its properties. We show that Green's relation $\mathbb{D}$ is a congruence relation on residuated skew lattice and its quotient algebra is a residuated lattice. Deductive system and skew deductive system in residuated skew lattices are defined and relationships between them are given and proved. We define branchwise residuated skew lattice and show that a conormal distributive residuated skew lattice is equivalent with a branchwise residuated skew lattice under a condition.

1 Introduction

The residuation is a fundamental concept of ordered structures. The operation $\odot : [0,1] \times [0,1] \to [0,1]$ which makes $([0,1], \odot, 1)$ a commutative monoid, i.e. $\odot$ is commutative, associative and $x \odot 1 = x$ that means it is a t-norm. If $\odot$ is a left-continuous t-norm, then putting $x \to y = sup\{z|z \odot x \leq y\}$ makes $([0,1], min, max, \odot, \to, 0, 1)$ a residuated lattice. If $\odot$ is a continuous t-norm, then $x \to y$ is called residuum such that $x \odot y \leq z$ iff $x \leq y \to z$. Residuated lattices were investigated by Krull (1924), then Ward and Dilworth wrote a series of important papers in this field. Apart from their interest in logic, residuated lattices have interesting algebraic properties and include two important classes of algebras: BL-algebra (introduced by Hajek as the algebraic counterpart of his basic logic) and MV-algebra (correspondent to Lukasiewicz

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many-valued logic) [18].
Skew lattices were introduced for the first time by Jordan [11]. Skew lattices are a generalization of lattices. A skew lattice is an algebra $(A, \vee, \wedge)$ such that $(A, \vee)$ and $(A, \wedge)$ are bands satisfying a variation of absorption laws. In skew lattice two different order concepts can be defined: the natural preorder, denoted by $\preceq$ and the natural partial order denoted by $\leq$, one weaker than the other and both of them motivated by analogous order concepts defined for bands. They generalize the partial order of the correspondent lattice. Though, unlike lattices, the admissible Hasse diagram representing the order structure of a skew lattice does not determine its algebraic structure, in general [19]. Green's relation $\mathbb{D}$ induced by the pre-order $\preceq$ is a congruence on $(A, \vee, \wedge)$ such that $(A/\mathbb{D}, \vee, \wedge)$ is a lattice too. Leech introduced skew Boolean algebras [16] and normal skew lattices [12]. Cvetko-Vah defined skew Heyting algebras as dual skew Boolean algebras [8].
Skew lattices are non-commutative generalization of lattices, so we provide non-commutative generalization of residuated lattices in this paper. By using residuum on skew lattices, we define residuated skew lattices as non-commutative generalization of residuated lattices and obtain properties of residuated skew lattices. The class of all conormal residuated skew lattices forms a variety under a condition. We show that Green's relation $\mathbb{D}$ is a congruence on residuated skew lattice A, and $A/\mathbb{D}$ is a residuated lattice. We define branchwise residuated skew lattices and show that a conormal distributive residuated skew lattice is equivalent with a branchwise residuated skew lattice under a condition. We define deductive system and skew deductive system in residuated skew lattices and give relationships between them.

2 Preliminaries

In this section, we review some properties of skew lattices which we need in the sequel.

Definition 2.1. *[2] A skew lattice is an algebra* $(A, \vee, \wedge)$ *of type* $(2, 2)$ *satisfying the following identities:*
(1) $(x \vee y) \vee z = x \vee (y \vee z)$ *and* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
(2) $x \wedge x = x$ *and* $x \vee x = x$,
(3) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ *and* $(x \wedge y) \vee y = y = (x \vee y) \wedge y$,
The identities found in $(1-3)$ *are known as the associative law, the idempotent laws and absorption laws respectively. In view of the associativity* (1), *we can omit parentheses when no ambiguity arises.*

On a given skew lattice A the natural partial order $\leq$ and natural preorder $\preceq$ respectively are defined by $x \leq y$ iff $x \wedge y = x = y \wedge x$ or dually

$x \vee y = y = y \vee x$ and $x \preceq y$ iff $y \vee x \vee y = y$ or equivalently $x \wedge y \wedge x = x$. Relation $\mathbb{D}$ is defined by $x \;\mathbb{D}\; y$ iff $x \vee y \vee x = x$ and $y \vee x \vee y = y$ or dually, $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. $\mathbb{D}$ is called the natural equivalence and it coincides with Green's relation $\mathbb{D}$ on both semigroups $(A, \wedge)$ and $(A, \vee)$ [8]. For elements x and y of a skew lattice A the following are equivalent [20]:

$$(1)\; x \leq y, \;\; (2)\; x \vee y \vee x = y, \;\; (3)\; y \wedge x \wedge y = x.$$

Leech's first decomposition theorem for skew lattices states that the relation $\mathbb{D}$ is a congruence on a skew lattice A, $A/\mathbb{D}$ is the maximal lattice image of A, and each congruence class is a maximal rectangular skew lattice in A [15]. A pair of natural congruences, $\mathbb{L}$ and $\mathbb{R}$, refine $\mathbb{D}$ [7]. We say that x is $\mathbb{L}$-related to y (denoted $x \,\mathbb{L}\, y$) if $x \wedge y = x$ and $y \wedge x = y$, or dually, $x \vee y = y$ and $y \vee x = x$. Likewise, x and y are $\mathbb{R}$-related ($x \,\mathbb{R}\, y$) if $x \wedge y = y$ and $y \wedge x = x$, or dually, $x \vee y = x$ and $y \vee x = y$. A skew lattice is left-handed if $\mathbb{D} = \mathbb{L}$ so that $x \wedge y = x = y \vee x$ on each rectangular subalgebra. Left-handed skew lattices are characterized by various equivalent identities:

$$x \wedge y \wedge x = x \wedge y \;\; \textit{or} \;\; x \wedge (y \vee x) = x \;\; \textit{or} \;\; x \vee y \vee x = y \vee x \;\; \textit{or} \;\; (x \wedge y) \vee x = x.$$

For instance, if $x \wedge y \wedge x = x \wedge y$ holds identically, then $x \wedge (y \vee x) = x \wedge (y \vee x) \wedge x = x \wedge x = x$. If $x \wedge (y \vee x) = x$ holds, then $y \vee x = (x \wedge (y \vee x)) \vee y \vee x = x \vee y \vee x$. Similar arguments show that the third identity implies the fourth and that the fourth implies the first. Dually a skew lattice is right-handed if $\mathbb{D} = \mathbb{R}$ so that $x \wedge y = y = y \vee x$ on each rectangular subalgebra. Right-handed skew lattices are characterized by the following equivalent identities:

$$x \wedge y \wedge x = y \wedge x \;\; \textit{or} \;\; (x \vee y) \wedge x = x \;\; \textit{or} \;\; x \vee y \vee x = x \vee y \;\; \textit{or} \;\; x \vee (y \wedge x) = x.$$

For any elements x, y of a skew lattice A, $x \,\mathbb{D}\, y$ iff $x \vee y = y \wedge x$. Also if $\mathbb{D}$ is a congruence and $A/\mathbb{D}$ is a lattice, for given any congruence $\mathbb{C}$ on A such that $A/\mathbb{C}$ is a lattice, $\mathbb{D} \subseteq \mathbb{C}$, then $A/\mathbb{D}$ is the maximal lattice image of A [7].

Proposition 2.1. *[15] Let B and C be comparable $\mathbb{D}$-classes in a skew lattice A such that $B \prec C$. For all $x, y \in A$, $x \leq y$ implies $x \preceq y$. Furthermore, whenever $x \in C, y \in B$, $B \preceq C$ iff $y \preceq x$.*

Definition 2.2. *[12] Skew lattice A is normal if $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$ and is conormal if $x \vee y \vee z \vee w = x \vee z \vee y \vee w$ for all $x, y, z, w \in A$.*

Proposition 2.2. *[12] A skew lattice A is normal iff each sub skew lattice $(\downarrow x)$ is a sub lattice of A. Dually, A is conormal iff each sub skew lattice $(\uparrow x)$ is a sub lattice of A $((\uparrow x) = \{y \in A | y \geq x\}, (\downarrow x) = \{y \in A | y \leq x\})$.*

A skew lattice is distributive if it satisfies $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$ [13].

A skew lattice A is quasi-distributive, if the maximal lattice image $A/\mathbb{D}$ is a distributive lattice. All distributive skew lattices are quasi-distributive [13].

Theorem 2.1. *[12] Given a skew lattice A, then the following are equivalent:*
(1) $x \vee (y \wedge z) \vee w = (x \vee y \vee w) \wedge (x \vee z \vee w)$ *holds on A.*
(2) *A is distributive and conormal.*
(3) *$A/\mathbb{D}$ is distributive and A is conormal.*

Lemma 2.1. *[8] Let A be a conormal skew lattice and let C and B be comparable $\mathbb{D}$-classes such that $B \preceq C$ holds in the lattice $A/\mathbb{D}$. Given $b \in B$ there exists a unique $a \in C$ such that $b \leq a$.*

It was proved in [15] that a skew lattice always forms a regular band for either of the operations $\wedge, \vee$, i.e $x \wedge y \wedge x \wedge z \wedge x = x \wedge y \wedge z \wedge x$ and $x \vee y \vee x \vee z \vee x = x \vee y \vee z \vee x$.
A skew chain is a skew lattice where $A/\mathbb{D}$ is a chain i.e. for all $x, y \in A$, $x \preceq y$ or $y \preceq x$ [8].

Definition 2.3. *[18] A residuated lattice is an algebra $A = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following:*
(1) $(A, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
(2) $(A, \odot, 1)$ *is a commutative monoid,*
(3) $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e. $z \leq x \rightarrow y$ iff $x \odot z \leq y$, for all $x, y, z \in A$.*

A generalized residuated lattice is an algebra $A = (A, \vee, \wedge, \rightarrow, \odot, 1)$ such that A is a residuated lattice without the bottom element. If it also has a bottom element, then it is a residuated lattice.

Lemma 2.2. *[1] If A is a residuated lattice, then $x \rightarrow y = (x \vee y) \rightarrow y$, for all $x, y \in A$.*

Definition 2.4. *[22] A function $f : A * A \rightarrow P(A)$, of the set $A * A$ into the set of all nonempty subsets of A, is called a hyperoperation.*

3 Residuated skew lattices

In this section, we want to extend the notion of residuated lattice. We apply the residuum on skew lattice and define residuated skew lattice. Then we study its properties.

Definition 3.1. *A residuated skew lattice is a nonempty set A with operations $\vee, \wedge, \odot$ and hyperoperation $\to$ and constant element 1 that satisfying the following:*
(1) $(A, \vee, \wedge, 1)$ is a skew lattice with top 1 (for all $x \in A$, $x \leq 1$),
(2) $(A, \odot, 1)$ is a commutative monoid,
(3) $\odot$ and $\to$ form an adjoint pair, i.e. $z \preceq x \to y$ iff $x \odot z \preceq y$, for all $x, y, z \in A$.

The relation between the pair of operations $\odot$ and $\to$ expressed by (3), is a special case of the law of residuation and for every $x, y \in A, x \to y = \sup\{z \in A | x \odot z \preceq y\}$. Supremum of a set in a pre-ordered set is not a unique element, $x \to y$ may be a $\mathbb{D}$-class. Two $\mathbb{D}$-classes have $\mathbb{D}$-relationship when all of their members have $\mathbb{D}$-relationship with each other. Relation $\preceq$ between two $\mathbb{D}$-classes is defined member to member (i.e. $B \preceq C$ iff $\forall c \in C, \forall b \in B, b \preceq c$). Also each of the $\vee, \wedge, \odot, \to$, between two $\mathbb{D}$-classes are defined member to member ($B \to C = \{b \to c | b \in B, c \in C\}$).

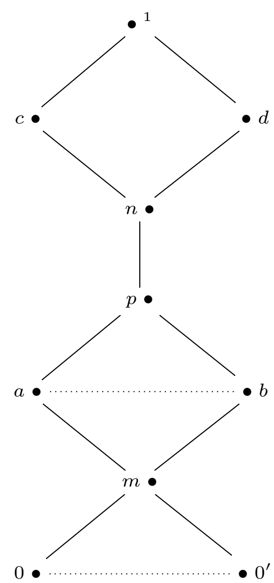
Example 3.1. *Let $A = \{0, 0', m, a, b, p, n, c, d, 1\}$ be a skew lattice such that $0', 0 < m < a, b < p < n < c, d < 1$, $0 \mathbin{\mathbb{D}} 0'$, $a \mathbb{D} b$ and $\mathbb{D}_a = \{a, b\}$, $\mathbb{D}_0 = \{0, 0'\}$. $A = (A, \vee, \wedge, \odot, \to, 1)$ is a residuated skew lattice with the following operations:*

$\to$	0	$0'$	m	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1	1	1
$0'$	1	1	1	1	1	1	1	1	1	1
m	$\mathbb{D}_0$	$\mathbb{D}_0$	1	1	1	1	1	1	1	1
a	$\mathbb{D}_0$	$\mathbb{D}_0$	$\mathbb{D}_a$	1	1	1	1	1	1	1
b	$\mathbb{D}_0$	$\mathbb{D}_0$	$\mathbb{D}_a$	1	1	1	1	1	1	1
p	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	1	1	1	1	1
n	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	1	1	1	1
c	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	d	1	d	1
d	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	c	c	1	1
1	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	n	c	d	1

$\odot$	0	$0'$	m	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0	0	0
$0'$	0	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$
m	0	$0'$	m	m	m	m	m	m	m	m
a	0	$0'$	m	m	m	a	a	a	a	a
b	0	$0'$	m	m	m	b	b	b	b	b
p	0	$0'$	m	a	b	p	p	p	p	p
n	0	$0'$	m	a	b	p	n	n	n	n
c	0	$0'$	m	a	b	p	n	c	n	c
d	0	$0'$	m	a	b	p	n	n	d	d
1	0	$0'$	m	a	b	p	n	c	d	1

$\vee$	0	$0'$	m	a	b	p	n	c	d	1
0	0	$0'$	m	a	b	p	n	c	d	1
$0'$	0	$0'$	m	a	b	p	n	c	d	1
m	m	m	m	a	b	p	n	c	d	1
a	a	a	a	a	b	p	n	c	d	1
b	b	b	b	a	b	p	n	c	d	1
p	p	p	p	p	p	p	n	c	d	1
n	n	n	n	n	n	n	n	c	d	1
c	c	c	c	c	c	c	c	c	1	1
d	d	d	d	d	d	d	d	1	d	1
1	1	1	1	1	1	1	1	1	1	1

$\wedge$	0	$0'$	m	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0	0	0
$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$
m	0	$0'$	m	m	m	m	m	m	m	m
a	0	$0'$	m	a	a	a	a	a	a	a
b	0	$0'$	m	b	b	b	b	b	b	b
p	0	$0'$	m	a	b	p	p	p	p	p
n	0	$0'$	m	a	b	p	n	n	n	n
c	0	$0'$	m	a	b	p	n	c	n	c
d	0	$0'$	m	a	b	p	n	n	d	d
1	0	$0'$	m	a	b	p	n	c	d	1



In Example 3.1, A is a residuated skew lattice but is not a generalized residuated lattice, because A is not lattice since $\vee, \wedge$ are not commutative ($a = b \vee a \neq a \vee b = b, b = b \wedge a \neq a \wedge b = a$).

Remark 3.1. *Any generalized residuated lattice is a residuated skew lattice but the converse is not true. Indeed, if $\vee, \wedge$ are commutative, then every residuated skew lattice is a generalized residuated lattice.*

From here until the end of this section, let A be a residuated skew lattice unless otherwise stated.

Lemma 3.1. *Let $x, y, z \in A$. Then*
(1) $1 \to x = \mathbb{D}_x$ *and* $x \to x = 1$, ($\mathbb{D}_x = \{y \in A | y \,\mathbb{D}\, x\}$),
(2) $x \odot y \preceq x, y$ *hence* $x \odot y \preceq x \wedge y, y \wedge x$, $y \preceq x \to y$,
(3) $x \odot y \preceq x \to y$,
(4) $x \preceq y$ *iff* $x \to y = 1$ *and* $x \,\mathbb{D}\, y$ *iff* $x \to y = y \to x = 1$,
(5) $x \to 1 = 1$,
(6) $x \odot (x \to y) \preceq y$, $x \preceq (x \to y) \to y$ *and* $(x \to y) \to y) \to y \,\mathbb{D}\, x \to y$,
(7) $x \to y \preceq x \odot z \to y \odot z$,
(8) $x \preceq y$ *implies* $x \odot z \preceq y \odot z$,
(9) $x \to y \preceq (z \to x) \to (z \to y)$,
(10) $x \to y \preceq (y \to z) \to (x \to z)$,
(11) $x \preceq y$ *implies* $z \to x \preceq z \to y$ *and* $y \to z \preceq x \to z$,

(12) $x \odot (y \to z) \preceq y \to (x \odot z) \preceq x \odot y \to x \odot z$,
(13) $x \to (y \to z) \,\mathbb{D}\, (x \odot y) \to z \,\mathbb{D}\, y \to (x \to z)$,
(14) $x_1 \to y_1 \preceq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$,
(15) $x \vee y \preceq (x \to y) \to y \wedge (y \to x) \to x \wedge (x \to y) \to y$,
(16) $x \odot (x \to y) \preceq (x \wedge y),\ (y \wedge x)$.

Proof. (1) We must show that $1 \to x = \sup\{z \in A | 1 \odot z \preceq x\} = \mathbb{D}_x = \{y \in A | y \,\mathbb{D}\, x\}$. Let $B = \{z \in A | 1 \odot z \preceq x\}$ and $t \in \mathbb{D}_x$. Then $t \,\mathbb{D}\, x$, therefore $x \preceq t$, thus t is an upper bound of B. Let $t' \in A$ be an upper bound of B i.e. $x \preceq t'$. Since $t \,\mathbb{D}\, x$, then $t \preceq t'$. Therefore $t \in \sup B$. Now, let $t \in \sup B$, then $t \preceq x$. Since $x \odot 1 = x \preceq x$, then $x \preceq 1 \to x$ i.e. $x \preceq \sup B$. Therefore $x \preceq t$, thus $t \,\mathbb{D}\, x$. Therefore $t \in \mathbb{D}_x$. And $1 \odot x = x \preceq x$ implies $1 \preceq x \to x$, therefore $(x \to x) = 1$.
(2) It is clear, by Proposition 3.1 of [5].
(3) Results of (1) and (2): $x \odot y \preceq y$ and $y \preceq x \to y$ so $x \odot y \preceq x \to y$.
(4) We have $x \preceq y$ iff $x \odot 1 \preceq y$ iff $1 \preceq x \to y$ iff $x \to y = 1$.
(5) Follows from (4).
(6) Concludes immediately from definition.
(7) $x \to y \preceq (x \odot z) \to (y \odot z)$ iff $(x \to y) \odot x \odot z \preceq y \odot z$ iff $(x \to y) \odot x \preceq z \to (y \odot z)$. But by (6), $(x \to y) \odot x \preceq y, y \preceq z \to (y \odot z)$ implies $(x \to y) \odot x \preceq z \to (y \odot z)$.
(8) Follows from (7).
(9) $x \to y \preceq (z \to x) \to (z \to y)$ iff $(x \to y) \odot (z \to x) \preceq z \to y$ iff $(x \to y) \odot (z \to x) \odot z \preceq y$. Which implies $(x \to y) \odot (z \to x) \odot z \preceq (x \to y) \odot x \preceq y$ by (6).
$(10 - 13)$ are clear, by Proposition 3.1 of [5].
(14) It is enough to prove that $(x_1 \to y_1) \odot (y_2 \to x_2) \odot (y_1 \to y_2) \odot x_1 \preceq x_2$, this is a consequence of applying several times (6).
(15) Since $x, y \preceq (x \to y) \to y$ and $x, y \preceq (y \to x) \to x$, then $x \vee y \preceq ((x \to y) \to y) \wedge ((y \to x) \to x) \wedge ((x \to y) \to y)$.
(16) It is clear by $(2), (6)$. $\square$

Theorem 3.1. *Relation* $\mathbb{D}$ ($x \preceq y, y \preceq x$ *iff* $x \,\mathbb{D}\, y$) *is a congruence relation on* A *and* $A/\mathbb{D}$ *is a residuated lattice.*

Proof. By Proposition 3.3 of [5] and Corollary 3.1 of [5], it is clear. $\square$

Definition 3.2. *A residuated skew lattice with* 0 *is a structure* $A = (A, \vee, \wedge, \odot, \to, 0, 1)$ *such that* $(A, \vee, \wedge, \odot, \to, 1)$ *is a residuated skew lattice and* $0 \in A$ *is a constant such that* $x \geq 0$*, for all* $x \in A$*.* 0 *is a unique element in a residuated skew lattice with* 0*.*

In a residuated skew lattice with 0, it makes sense to define a new operation as $x^* = x \to 0$.

Theorem 3.2. *Let A be a residuated skew lattice with 0 and $x, y \in A$. Then*
(1) $x \odot x^* = 0$, $x \odot y = 0$ *iff* $x \preceq y^*$,
(2) $0 \to x = 1$,
(3) $x \preceq x^{**}$ *and* $x^{**} \preceq x^* \to x$,
(4) $1^* = 0, 0^* = 1$,
(5) $x \preceq y$ *implies* $y^* \preceq x^*$,
(6) $x \to y \preceq y^* \to x^*$,
(7) $x^* \odot y^* \preceq (x \odot y)^*$,
(8) $x^{***} \,\mathbb{D}\, x^*, (x \odot y)^* \,\mathbb{D}\, x \to y^* \,\mathbb{D}\, y \to x^* \,\mathbb{D}\, x^{**} \to y^*$,
(9) $x \odot y = 0$ *iff* $x^{**} \odot y^{**} = 0$,
(10) $y \,\mathbb{D}\, (x^* \to x) \to x$ *implies* $y^* \preceq y$.

Proof. (1) By Proposition 4.1 of [5], it is clear.
(2) By Proposition 4.2 of [5], it is clear.
(3) By Proposition 4.3 of [5] we have $x \preceq x^{**}$. $x^{**} \to (x^* \to x) \,\mathbb{D}\, (x^{**} \odot x^*) \to x = 0 \to x = 1$.
(4) By Proposition 4.1 of [5], it is clear.
(5) By Lemma 3.1 (11), it is clear.
(6) It is clear, by Proposition 4.3 of [5].
(7) Since $x \odot y \preceq x, y$, then $x^*, y^* \preceq (x \odot y)^*$, it is clear.
(8) By Propositions 3.1, 4.3 of [5], It is clear.
(9) Since $x \odot y \preceq x^{**} \odot y^{**}$, one direction is clear. Now, $x \odot y = 0$ iff $x \preceq y^*$ implies $y^{**} \preceq x^*$ implies $x^{**} \preceq y^{***}$ iff $x^{**} \odot y^{**} = 0$ and proof is complete.
(10) Since $x, x^* \preceq y$, then $y^* \preceq x^*$ and $1 = y^* \to x^* \preceq y^* \to y$ implies $y^* \to y = 1$ implies $y^* \preceq y$. □

Lemma 3.2. *Let A be a residuated skew lattice with 0. Then for every $x, y \in A$, we have $x^{**} \odot y^{**} \preceq (x \odot y)^{**}$.*

Proof. By $(x \odot y)^* \,\mathbb{D}\, x \to y^*$, so $(x \odot y)^* \odot x \preceq y^*$. By Lemma 3.1 (11), we deduce that $y^{**} \preceq [(x \odot y)^* \odot x]^* \,\mathbb{D}\, [(x \odot y)^* \to x^*]$, so $y^{**} \odot (x \odot y)^* \preceq x^*$. Then $x^{**} \preceq [y^{**} \odot (x \odot y)^*]^* \,\mathbb{D}\, [y^{**} \to (x \odot y)^{**}]$, that is, $x^{**} \odot y^{**} \preceq (x \odot y)^{**}$. □

Corollary 3.1. *Let A be a residuated skew lattice with 0. Then for every $x \in A$ and $n \geq 1$ we have $(x^{**})^n \preceq (x^n)^{**}$, $((x^{**})^n = x^{**} \odot \cdots \odot x^{**})$.*

Theorem 3.3. *Let A be a residuated skew lattice with 0. Then for every $x \in A$, the following conditions are equivalent:*
(1) $x \to x^* \,\mathbb{D}\, x^*$,
(2) $(x^2)^* \,\mathbb{D}\, x^*$,

(3) $(x^n)^* \,\mathbb{D}\, x^*$,
(4) $x \odot (x \to x^*) = 0$.

Proof. $(1 \Rightarrow 2)$ $x^* \,\mathbb{D}\, x \to x^* = x \to (x \to 0) \,\mathbb{D}\, x^2 \to 0 = (x^2)^*$ by Lemma 3.1.
$(2 \Rightarrow 3)$ By Lemma 3.1, we have $(x^3)^* = (x^2 \odot x) \to 0 \;\mathbb{D}\; x \to (x^2 \to 0) \,\mathbb{D}\, x \to x^* = x \to (x \to 0) \,\mathbb{D}\, x^2 \to 0 = (x^2)^* \,\mathbb{D}\, x^*$. Similarly, $(x^n)^* \,\mathbb{D}\, x^*$.
$(3 \Rightarrow 4)$ By Lemma 3.1, we have $x \odot (x \to x^*) \,\mathbb{D}\, x \odot (x \to (x^n)^*) \,\mathbb{D}\, x \odot (x^{n+1} \to 0) = x \odot (x^{n+1})^* \,\mathbb{D}\, x \odot x^* = 0$.
$(4 \Rightarrow 1)$ $x \odot (x \to x^*) = 0$ implies $x \to x^* \preceq x^*$. On the other hand $x^* \preceq x \to x^*$, therefore $x \to x^* \,\mathbb{D}\, x^*$ by Lemma 3.1. □

Theorem 3.4. *Let $x, y_1, y_2 \in A$. Then*
(1) $x \odot (y_1 \vee y_2) \,\mathbb{D}\, (x \odot y_1) \vee (x \odot y_2)$,
(2) $x \odot (y_1 \wedge y_2) \preceq (x \odot y_1) \wedge (x \odot y_2)$,
(3) $x \to (y_1 \wedge y_2) \,\mathbb{D}\, (x \to y_1) \wedge (x \to y_2)$,
(4) $(y_1 \vee y_2) \to x \,\mathbb{D}\, (y_1 \to x) \wedge (y_2 \to x)$,
(5) $(y_1 \to x) \vee (y_2 \to x) \preceq (y_1 \wedge y_2) \to x$,
(6) $(x \to y_1) \vee (x \to y_2) \preceq x \to (y_1 \vee y_2)$,
(7) $(y_1 \vee y_2)^* \,\mathbb{D}\, y_1^* \wedge y_2^*$,
(8) $(y_1 \wedge y_2)^* \succeq y_1^* \vee y_2^*$.
Parts $7, 8$ *are true in a residuated skew lattice with* 0.

Proof. (1) Clearly $x \odot y_i \preceq x \odot (y_1 \vee y_2)$ for every $(i = 1, 2)$. Therefore $(x \odot y_1) \vee (x \odot y_2) \preceq x \odot (y_1 \vee y_2)$.
Conversely, since for every $(i = 1, 2)$ $x \odot y_i \preceq (x \odot y_1) \vee (x \odot y_2)$, then $y_i \preceq x \to (x \odot y_1) \vee (x \odot y_2)$. Which implies $y_1 \vee y_2 \preceq x \to (x \odot y_1) \vee (x \odot y_2)$. Therefore $x \odot (y_1 \vee y_2) \preceq (x \odot y_1) \vee (x \odot y_2)$. So we obtain $x \odot (y_1 \vee y_2) \mathbb{D} (x \odot y_1) \vee (x \odot y_2)$.
(2) Since $y_1 \wedge y_2 \preceq y_1, y_2$, it is clear.
(3) Let $y = y_1 \wedge y_2$. Then $y \preceq y_1, y_2$, for every $(i = 1, 2)$, we deduce that $x \to y \preceq x \to y_i$, hence $x \to y \preceq (x \to y_1) \wedge (x \to y_2)$. On the other hand $(x \to y_1) \wedge (x \to y_2) \preceq x \to y$ is equivalent with $x \odot ((x \to y_1) \wedge (x \to y_2)) \preceq y$. This is true because by (2) we have $x \odot ((x \to y_1) \wedge (x \to y_2)) \preceq (x \odot (x \to y_1)) \wedge (x \odot (x \to y_2)) \preceq y_1 \wedge y_2 = y$.
(4) Let $y = y_1 \vee y_2$. Since for every $(i = 1, 2)$, $y_i \preceq y$, then $y \to x \preceq y_i \to x$. Therefore $y \to x \preceq (y_1 \to x) \wedge (y_2 \to x)$.
Conversely, $(y_1 \to x) \wedge (y_2 \to x) \preceq y \to x$ iff $y \odot ((y_1 \to x) \wedge (y_2 \to x)) \preceq x$. By (2), (1), we have $y \odot ((y_1 \to x) \wedge (y_2 \to x)) \preceq (y \odot (y_1 \to x)) \wedge (y \odot (y_2 \to x)) \,\mathbb{D}\, (y_1 \odot (y_1 \to x) \vee y_2 \odot (y_1 \to x)) \wedge ((y_1 \odot (y_2 \to x)) \vee (y_2 \odot (y_2 \to x)) \preceq x \wedge x = x$. So we obtain $(y_1 \vee y_2) \to x \,\mathbb{D}\, (y_1 \to x) \wedge (y_2 \to x)$.
(5) By Lemma 3.1 (11), for every $(i = 1, 2), y_i \to x \preceq (y_1 \wedge y_2) \to x$.
(6) Is similar to (5).

(7) By taking $x = 0$ in (4), we obtain $(y_1 \vee y_2)^* \, \mathbb{D} \, (y_1^* \wedge y_2^*)$.
(8) By taking $x = 0$ in (5), we obtain $(y_1 \wedge y_2)^* \succeq y_1^* \vee y_2^*$. □

Corollary 3.2. *If $x, x_1, y, y_1, z \in A$, then*
(1) $x \vee y = 1$ *implies* $(x \odot y) \, \mathbb{D} \, (x \wedge y)$,
(2) $x \to (y \to z) \succeq (x \to y) \to (x \to z)$,
(3) *If A is conormal, then* $x \vee (y \odot z) \vee x \succeq (x \vee y \vee x) \odot (x \vee z \vee x)$, *hence* $x \vee y^n \vee x \succeq (x \vee y \vee x)^n$ *and* $x^m \vee y^n \vee x^m \succeq (x \vee y \vee x)^{mn}$, *for any m, n natural numbers,*
(4) $(x \to y) \odot (x_1 \to y_1) \preceq (x \vee x_1) \to (y \vee y_1)$,
(5) $(x \to y) \odot (x_1 \to y_1) \preceq (x \wedge x_1) \to (y \wedge y_1)$.

Proof. (1) Suppose $x \vee y = 1$. Clearly $x \odot y \preceq x$ and $x \odot y \preceq y$. Let $t \in A$ be such that $t \preceq x$ and $t \preceq y$. By Lemma 3.1 (12), we have $t \to (x \odot y) \succeq x \odot (t \to y) = x \odot 1 = x$ and $t \to (x \odot y) \succeq y \odot (t \to x) = y \odot 1 = y$, so $t \to (x \odot y) \succeq x \vee y = 1$, hence $t \to (x \odot y) = 1$ iff $t \preceq x \odot y$, that is $(x \odot y) \, \mathbb{D} \, (x \wedge y)$.
(2) We have by Lemma 3.1 (13): $(x \to (y \to z)) \, \mathbb{D} \, ((x \odot y) \to z)$ and $((x \to y) \to (x \to z)) \, \mathbb{D} \, [x \odot (x \to y)] \to z$. But $x \odot y \preceq x \odot (x \to y)$, so we obtain $(x \odot y) \to z \succeq [x \odot (x \to y)] \to z$ iff $x \to (y \to z) \succeq (x \to y) \to (x \to z)$.
(3) By Theorem 3.4 (1) and by assumption we deduce

$$
\begin{aligned}
(x \vee y \vee x) \odot (x \vee z \vee x) \, \mathbb{D} \, & (x^2 \vee (x \odot y) \vee x^2 \vee (x \odot z) \vee (y \odot z) \vee (x \odot z) \vee x^2 \vee (x \odot y) \vee x^2) \\
& \preceq (x \vee (y \odot z) \vee (x \odot z) \vee x^2 \vee (x \odot y) \vee x^2) \\
& = (x \vee (x \odot z) \vee (y \odot z) \vee x^2 \vee (x \odot y) \vee x^2) \\
& \preceq (x \vee (y \odot z) \vee x^2 \vee (x \odot y) \vee x^2) \\
& \preceq (x \vee (y \odot z) \vee (x \odot y) \vee x^2) \\
& \preceq (x \vee (y \odot z) \vee x^2) \\
& \preceq (x \vee (y \odot z) \vee x).
\end{aligned}
$$

(4) From:

$$x \odot (x \to y) \odot (x_1 \to y_1) \preceq x \odot (x \to y) \preceq x \wedge y \preceq y \vee y_1$$

and

$$x_1 \odot (x \to y) \odot (x_1 \to y_1) \preceq x_1 \odot (x_1 \to y_1) \preceq x_1 \wedge y_1 \preceq y \vee y_1$$

we deduce that
$(x \to y) \odot (x_1 \to y_1) \preceq x \to (y \vee y_1)$ and $(x \to y) \odot (x_1 \to y_1) \preceq x_1 \to (y \vee y_1)$.
So by Theorem 3.4 (4),

$(x \to y) \odot (x_1 \to y_1) \preceq ([x \to (y \vee y_1)] \wedge [x_1 \to (y \vee y_1)]) \, \mathbb{D} \, ((x \vee x_1) \to (y \vee y_1))$.
(5) Lemma 3.1 (6) implies

$$(x \wedge x_1) \odot (x \to y) \odot (x_1 \to y_1) \preceq x \odot (x \to y) \preceq y$$

and

$$(x \wedge x_1) \odot (x \to y) \odot (x_1 \to y_1) \preceq x_1 \odot (x_1 \to y_1) \preceq y_1$$

so we deduce that

$$(x \to y) \odot (x_1 \to y_1) \preceq (x \wedge x_1) \to y$$

and

$$(x \to y) \odot (x_1 \to y_1) \preceq (x \wedge x_1) \to y_1.$$

So by Theorem 3.4

$(x \to y) \odot (x_1 \to y_1) \preceq [[(x \wedge x_1) \to y] \wedge [(x \wedge x_1) \to y_1]] \, \mathbb{D} \, [(x \wedge x_1) \to (y \wedge y_1)]$.

□

4 (Skew) deductive systems in residuated skew lattices

From here until the end of this section, let A be a residuated skew lattice unless otherwise stated.

Definition 4.1. *A nonempty subset $D \subseteq A$ is called a deductive system (for short ds) of A, if the following conditions are satisfied:*
(1) *$1 \in D$,*
(2) *If $x \in D, x \to y \subseteq D$, then $y \in D$ (If $x \to y$ is a single element and is not a $\mathbb{D}$-class we write $x \to y \in D$ instead $x \to y \subseteq D$).*

Example 4.1. *In Example 3.1, $D = \{p, n, c, d, 1\}$ is a ds.*

Remark 4.1. (1) *A ds D is proper iff no bottom element belong A. If A is a residuated skew lattice with 0, then D is a proper ds iff $0 \notin D$ iff no element $x \in A$ holds $x \in D, x^* \subseteq D$,*
(2) *$x \in D$ iff $x^n \in D$ for every $n \geq 1$,*
(3) *If $x \in D, x \, \mathbb{D} \, y$, then $y \in D$.*

Proposition 4.1. *A nonempty subset $D \subseteq A$ is a ds of A, iff for all $x, y \in A$ the following conditions are satisfied:*
(1′) *If $x \in D$ and $x \preceq y$, then $y \in D$,*
(2′) *If $x, y \in D$, then $x \odot y \in D$.*

Proof. Let D be a deductive system of A and $x \preceq y, x \in D$. So $x \to y = 1 \in D$ therefore $y \in D$. Now, let $x, y \in D$. We must show that $x \odot y \in D$. Since $x \odot y \to x \odot y = 1 \in D$, $x, y \in D$ and $x \odot y \to x \odot y \,\mathbb{D}\, x \to (y \to x \odot y)$ therefore $(x \to (y \to x \odot y)) \subseteq D$. Thus we deduce $x \odot y \in D$.
Conversely, let $(1'), (2')$ be satisfied. We must show that D is a ds. Let $x \preceq 1$ and $x \in D$ by assumption, $1 \in D$. Now, let $x \in D, x \to y \subseteq D$. Thus $x \odot (x \to y) \subseteq D$ and since $x \odot (x \to y) \preceq y$ we get that $y \in D$. □

Definition 4.2. *A nonempty subset $D \subseteq A$ is called a skew deductive system of A, if the following conditions are satisfied:*
(1) $1 \in D$,
(2) *If* $x \in D, x \to y \subseteq D$, *then* $(y \wedge (x \to y) \wedge y) \subseteq D$.

Lemma 4.1. *Any deductive system in A is a skew deductive system.*

Proof. Let D be a deductive system in A and $x \in D, x \to y \subseteq D$. Now we show $(y \wedge (x \to y) \wedge y) \subseteq D$. Since D is a deductive system and $x \in D, x \to y \subseteq D$, then $y \in D$. Therefore by Proposition 4.1, $y \odot (x \to y) \odot y \subseteq D$ so $(y \wedge (x \to y) \wedge y) \subseteq D$. □

Example 4.2. $F = \{n, c, d, 1\}$ *of Example 3.1, is a ds and also a skew ds.*

By bi-residuum on a residuated skew lattice A we understand the derived operation $\leftrightarrow$ defined for $x, y \in A$ by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ or $x \leftrightarrow y = (y \to x) \wedge (x \to y)$ that $(x \to y) \wedge (y \to x) \,\mathbb{D}\, (y \to x) \wedge (x \to y)$, in fact, $x \leftrightarrow y$ may be a $\mathbb{D}$-class.

Theorem 4.1. *Let* $x, y, x_1, x_2, y_1, y_2 \in A$. *Then*
(1) $x \leftrightarrow 1 = \mathbb{D}_x$,
(2) $x \leftrightarrow y \,\mathbb{D}\, y \leftrightarrow x$,
(3) $x \leftrightarrow y = 1$ *iff* $x \,\mathbb{D}\, y$,
(4) $(x \leftrightarrow y) \odot (y \leftrightarrow z) \preceq (x \leftrightarrow z)$,
(5) $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \preceq (x_1 \wedge x_2) \leftrightarrow (y_1 \wedge y_2)$,
(6) $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \preceq (x_1 \vee x_2) \leftrightarrow (y_1 \vee y_2)$,
(7) $(x_1 \leftrightarrow y_1) \odot (x_2 \leftrightarrow y_2) \preceq (x_1 \odot x_2) \leftrightarrow (y_1 \odot y_2)$,
(8) $(x_1 \leftrightarrow y_1) \odot (x_2 \leftrightarrow y_2) \preceq (x_1 \leftrightarrow x_2) \leftrightarrow (y_1 \leftrightarrow y_2)$.

Proof. $(1, 2, 3)$ are immediate consequences of Lemma 3.1.
(4) By Lemma 3.1 (10), $(x \to y) \odot (y \to z) \preceq x \to z$, therefore $(x \leftrightarrow y) \odot (y \leftrightarrow z) \preceq (x \to y) \odot (y \to z) \preceq x \to z$. Similarly, $(x \leftrightarrow y) \odot (y \leftrightarrow z) \preceq z \to x$. We

conclude that $(x \leftrightarrow y) \odot (y \leftrightarrow z) \preceq x \leftrightarrow z$.
(5) If we denote $a = x_1 \leftrightarrow y_1$ and $b = x_2 \leftrightarrow y_2$, then

$$\begin{aligned}(a \wedge b) \odot (x_1 \wedge x_2) &\preceq [(x_1 \to y_1) \wedge (x_2 \to y_2)] \odot (x_1 \wedge x_2)\\ &\preceq [(x_1 \to y_1) \odot x_1] \wedge [(x_2 \to y_2) \odot x_2]\\ &\preceq y_1 \wedge y_2.\end{aligned}$$

Which implies $a \wedge b \preceq (x_1 \wedge x_2) \to (y_1 \wedge y_2)$. Analogously we deduce that $a \wedge b \preceq (y_1 \wedge y_2) \to (x_1 \wedge x_2)$, hence $a \wedge b \preceq (x_1 \wedge x_2) \leftrightarrow (y_1 \wedge y_2)$.
(6) With the notations of (5) we have

$$\begin{aligned}(a \wedge b) \odot (x_1 \vee x_2) &\mathbb{D}\, [(a \wedge b) \odot x_1] \vee [(a \wedge b) \odot x_2]\\ &\preceq [(x_1 \to y_1) \odot x_1] \vee [(x_2 \to y_2) \odot x_2]\\ &\preceq y_1 \vee y_2.\end{aligned}$$

Which implies $a \wedge b \preceq (x_1 \vee x_2) \to (y_1 \vee y_2)$. Analogously we deduce that $a \wedge b \preceq (y_1 \vee y_2) \to (x_1 \vee x_2)$, hence $a \wedge b \preceq (x_1 \vee x_2) \leftrightarrow (y_1 \vee y_2)$.
(7) Consider

$$\begin{aligned}(a \odot b) \odot (x_1 \odot x_2) &\preceq [(x_1 \to y_1) \odot x_1] \odot [(x_2 \to y_2) \odot x_2]\\ &\preceq y_1 \odot y_2.\end{aligned}$$

then $a \odot b \preceq (x_1 \odot x_2) \to (y_1 \odot y_2)$. Hence analogously we deduce that $a \odot b \preceq (y_1 \odot y_2) \to (x_1 \odot x_2)$, so $a \odot b \preceq (x_1 \odot x_2) \leftrightarrow (y_1 \odot y_2)$.
(8) Consider

$$\begin{aligned}(a \odot b) \odot (x_1 \to x_2) &\preceq (y_1 \to x_1) \odot (x_2 \to y_2) \odot (x_1 \to x_2)\\ &\preceq (y_1 \to x_2) \odot (x_2 \to y_2)\\ &\preceq y_1 \to y_2.\end{aligned}$$

The proof is similar to the proof of (7). □

Proposition 4.2. *Let $x, y_1, y_2, z_1, z_2 \in A$. If $x \preceq y_1 \leftrightarrow y_2$ and $x \preceq z_1 \leftrightarrow z_2$, then $x^2 \preceq (y_1 \leftrightarrow z_1) \leftrightarrow (y_2 \leftrightarrow z_2)$.*

Proof. From $x \preceq y_1 \leftrightarrow y_2$ implies $x \preceq y_2 \to y_1$, thus $x \odot y_2 \preceq y_1$ and analogously we deduce that $x \odot z_1 \preceq z_2$. Then $x \odot x \preceq (y_1 \to z_1) \to (y_2 \to z_2)$ iff $x \odot x \odot (y_1 \to z_1) \preceq (y_2 \to z_2)$ iff $x \odot x \odot (y_1 \to z_1) \odot y_2 \preceq z_2$. Indeed $x \odot x \odot (y_1 \to z_1) \odot y_2 \preceq x \odot (y_1 \to z_1) \odot y_1 \preceq x \odot z_1 \preceq z_2$ and analogously $x \odot x \preceq (y_2 \to z_2) \to (y_1 \to z_1)$. □

Proposition 4.3. *Suppose $x, x_i, y_i \in A$, $(i = 1, 2)$. If $x \preceq x_i \leftrightarrow y_i$ for every $(i = 1, 2)$, then $x \preceq (x_1 \wedge x_2) \leftrightarrow (y_1 \wedge y_2)$.*

Proof. Since $x \preceq x_i \leftrightarrow y_i$ for every $i = 1, 2$, we deduce that $x \odot x_i \preceq y_i$ and then $x \odot (x_1 \wedge x_2) \preceq (x \odot x_1) \wedge (x \odot x_2) \preceq y_1 \wedge y_2$, hence $x \preceq (x_1 \wedge x_2) \to (y_1 \wedge y_2)$. Analogously, $x \preceq (y_1 \wedge y_2) \to (x_1 \wedge x_2)$. □

We denote by $Ds(A)$ the set of all deductive systems of A.
We study connections between the congruences of A and the deductive systems of A. For any deductive systems D of A we can associate a congruence Θ_D on A by
$(x, y) \in \Theta_D$ iff $x \to y, y \to x \subseteq D$ iff $(x \to y) \odot (y \to x) \subseteq D$.
Conversely, for $\Theta \in Con(A)$, the subset D_Θ of A defined by $x \in D_\Theta$ iff $(x, 1) \in \Theta$ is a deductive system of A. Moreover the natural maps associated with the above are mutually inverse and establish an isomorphism between $Ds(A)$ and $Con(A)$. So, we have the following result:

Theorem 4.2. *If $D \in Ds(A)$ and $\Theta \in Con(A)$, then*
(1) $\Theta_D \in Con(A)$ and $D_\Theta \in Ds(A)$,
(2) The assignments $D \longrightarrow \Theta_D$ and $\Theta \longrightarrow D_\Theta$ give a isomorphism between $Ds(A)$ and $Con(A)$.

Proof. (1) Clearly Θ_D is equivalence relation. We must show that Θ_D preserve operations. Let $x_1, x_2, y_1, y_2 \in A$ and $(x_1, y_1) \in \Theta_D$ and $(x_2, y_2) \in \Theta_D$. We must show that $(x_1 \wedge x_2)\ \Theta_D\ (y_1 \wedge y_2)$, $(x_1 \vee x_2)\ \Theta_D\ (y_1 \vee y_2)$ and $(x_1 \odot x_2)\ \Theta_D\ (y_1 \odot y_2)$ and $(x_1 \to x_2)\ \Theta_D\ (y_1 \to y_2)$.We only prove that Θ_D preserve $\wedge$. Since $(x_1, y_1) \in \Theta_D$ and $(x_2, y_2) \in \Theta_D$, then we have $x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2 \subseteq D$. Therefore by Theorem 4.1, we have $(x_1 \wedge x_2)\ \Theta_D\ (y_1 \wedge y_2)$.
Conversely, let Θ_D be congruence relation. We must show that D_Θ is a deductive system. Since $(1, 1) \in \Theta_D$, then $1 \in D_\Theta$. Let $x \in D_\Theta$ and $x \to y \subseteq D_\Theta$. Since $x \in D_\Theta$ implies $x\ \Theta_D\ 1$, on the other hand $y\ \Theta_D\ y$ therefore by assumption $(x \to y)\Theta_D(1 \to y)$ which implies $x \to y\ \Theta_D\ \mathbb{D}_y$. Also $(x \to y) \subseteq D_\Theta$ implies $(x \to y)\ \Theta_D\ 1$ therefore $1\ \Theta_D\ \mathbb{D}_y$. It implies $\mathbb{D}_y \subseteq D_\Theta$, therefore $y \in D_\Theta$.
(2) Let $\varphi : Ds(A) \longrightarrow Con(A)$ be such that $D \longmapsto \Theta_D$. It is enough to show that φ is surjective and order embedding. Let $\Theta_D \in Con(A)$ and define $D := \{x \in A | (x, 1) \in \Theta_D\}$. By (1), $D \in Ds(A)$ therefore φ is surjective. No, we must show that $D_1 \subseteq D_2$ iff $\Theta_{D_1} \subseteq \Theta_{D_2}$. Suppose $D_1 \subseteq D_2$ and $(x, y) \in \Theta_{D_1}$ therefore $x \to y, y \to x \subseteq D_1 \subseteq D_2$. This implies $(x, y) \in \Theta_{D_2}$. If $x \in D_1$, then $(x, 1) \in \Theta_{D_1}$. Which implies $(x, 1) \in \Theta_{D_2}$ i.e. $x \in D_2$. So φ is a isomorphism. □

If D is a deductive system of A, then we define relation $\preceq$ on A/D as $x/D \preceq y/D$ iff $x \to y \subseteq D$, $(A/D = \{x/D|x \in A\}$ and $x/D = \{y \in A|x\ \Theta_D\ y\})$.

Proposition 4.4. *Let $D \in Ds(A)$ and $x, y \in A$. Then*
(1) $x/D\ \mathbb{D}\ 1/D$ iff $\mathbb{D}_x \subseteq D$, hence $x/D\ \not{\mathbb{D}}\ 1/D$ iff $\mathbb{D}_x \nsubseteq D$,
(2) $x/D\ \mathbb{D}\ 0/D$ iff $x^* \subseteq D$,
(3) If D is proper and $x/D\ \mathbb{D}\ 0/D$, then $x \notin D$.
Parts $2, 3$ are true in a residuated skew lattice with 0.

Proof. (1) We have $x/D\ \mathbb{D}\ 1/D$ iff $(x \to 1) \odot (1 \to x) \subseteq D$ iff $1 \odot \mathbb{D}_x = \mathbb{D}_x \subseteq D$.
(2) We have $x/D\ \mathbb{D}\ 0/D$ iff $(x \to 0) \odot (0 \to x) \subseteq D$ iff $x^* \odot 1 = x^* \subseteq D$.
(3) Follows from Remark 4.1.

□

For a nonempty subset $S \subseteq A$, the smallest ds of A which contains S, i.e. $\cap\{D \in Ds(A)|S \subseteq D\}$, is said to be the ds of A generated by S and will be denoted by $[S)$. If $S = \{x\}$, with $x \in A$, we denote by $[x)$ the ds generated by $\{x\}$ ($[x)$ is called principal). For $D \in Ds(A)$ and $x \in A$, we denote by $D(x) = [D \cup \{x\})$ (clearly, if $x \in D$, then $D(x) = D$).

Proposition 4.5. *Let $S \subseteq A$ be a nonempty subset of A, $x \in A$ and $D, D_1, D_2 \in Ds(A)$. Then*
(1) If S is a deductive system, then $[S) = S$,
(2) $[S) = \{y \in A|s_1 \odot \ldots \odot s_n \preceq y$, for some $n \geq 1$ and $s_1, \ldots, s_n \in S\}$. In particular, $[x) = \{y \in A|y \succeq x^n$, for some $n \geq 1\}$,
(3) $D(x) = \{y \in A|y \succeq d \odot x^n$, which $d \in D$ and $n \geq 1\}$,
(4) $[D_1 \cup D_2) = \{y \in A|y \succeq d_1 \odot d_2$ for some $d_1 \in D_1$ and $d_2 \in D_2\}$.

Proof. (1) It is clear.
(2) Let $S' = \{y \in A|s_1 \odot \ldots \odot s_n \preceq y$, for some $n \geq 1$ and $s_1, \ldots, s_n \in S\}$. It is clear that S' is a deductive system which contains the set S, hence $[S) \subseteq S'$. Let $D \in Ds(A)$ such that $S \subseteq D$ and $y \in S'$. Then there exist $s_1, \ldots, s_n \in S$ such that $s_1 \odot \ldots \odot s_n \preceq y$. Since $s_1, \ldots, s_n \in D$, then $s_1 \odot \ldots \odot s_n \in D$, which implies $y \in D$, hence $S' \subseteq D$, we deduce that $S' \subseteq \cap D = [S)$, that is, $[S) = S'$.
$(3), (4)$ prove by (2). □

Definition 4.3. *A ds of A is maximal if it is proper and it is not contained in any other proper ds.*

Example 4.3. *In Example 3.1, $D = \{m, a, b, p, n, c, d, 1\}$ is a maximal ds.*

Theorem 4.3. *Let A be a residuated skew lattice with 0. If D is a proper ds of A, then the following conditions are equivalent:*
(1) *D is a maximal ds,*
(2) *For any $x \notin D$ there exist $d \in D, n \geq 1$ such that $d \odot x^n = 0$.*

Proof. ($1 \Rightarrow 2$) If $x \notin D$, then $[D \cup x) = A$, hence $0 \in [D \cup x)$. Therefore by Proposition 4.5, there exist $n \geq 1$ and $d \in D$ such that $d \odot x^n \preceq 0$. Thus $d \odot x^n = 0$.
($2 \Rightarrow 1$) Assume that there is a proper ds D_1 such that $D \subset D_1$. Then there exists $x \in D_1$ such that $x \notin D$. By hypothesis there exist $d \in D, n \geq 1$ such that $d \odot x^n = 0$. But $x, d \in D_1$ hence we obtain $0 \in D_1$, which is a contradiction. □

Corollary 4.1. *Let A be a residuated skew lattice with 0. If M is a proper ds of A, then the following are equivalent:*
(1) *M is a maximal ds,*
(2) *For any $x \in A, x \notin M$ iff $(x^n)^* \subseteq M$, for some $n \geq 1$.*

5 Branchwise residuated skew lattices

Now, we consider branches in a skew lattice and clearly each of the branches in a conormal skew lattice is a lattice. We want to define the residuum in branches and study its properties.

Definition 5.1. *A branchwise residuated skew lattice is an algebra $A = (A, \vee, \wedge, \odot, \rightarrow, 1)$ of type $(2, 2, 2, 2, 0)$ satisfying the following:*
(1) *$(A, \vee, \wedge, 1)$ is a distributive skew lattice with top 1 (for all $x \in A, x \leq 1$),*
(2) *$(A, \odot, 1)$ is a commutative monoid,*
(3) *For any $u \in A$, two operations $\rightarrow_u, \odot_u$ can be defined on $(\uparrow u)$ such that $(\uparrow u, \vee, \wedge, \odot_u, \rightarrow_u, u, 1)$ is a distributive residuated lattice by top 1 and bottom u,*
(4) *$x \rightarrow y = (y \vee x \vee y) \rightarrow_y y$,*
(5) *$x \odot y \, \mathbb{D} \, x \odot_u y$, for every $u \in A$, $x, y \in (\uparrow u)$.*

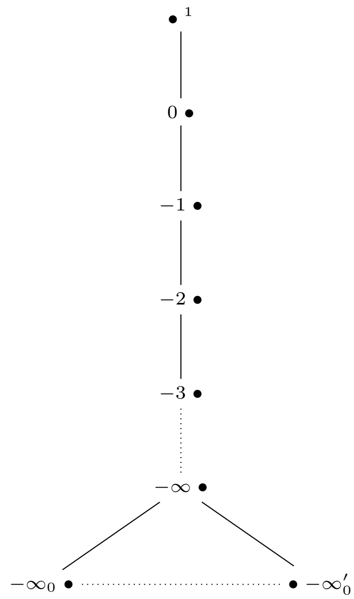
Lemma 5.1. *Let A be a (distributive) skew lattice such that $(\uparrow u)$ for every $u \in A$ be a (distributive) residuated lattice and $x \rightarrow y = y \vee x \vee y \rightarrow_y y$, for $x, y \in (\uparrow u)$. Then $x \rightarrow y = x \rightarrow_u y$.*

Proof. Since $x, y \in (\uparrow u)$, then $x \vee y \in (\uparrow u)$. Thus by Lemma 2.2, assumption and since $x \vee y \rightarrow_y y, x \vee y \rightarrow_u y \in (\uparrow y)$ we have $x \rightarrow y = y \vee x \vee y \rightarrow_y y = x \vee y \rightarrow_y y = x \vee y \rightarrow_u y = x \rightarrow_u y$. □

Example 5.1. *Let* $A = \{\mathbb{D}_0, -\infty, ... -3, -2, -1, 0, 1\}$ *be a skew lattice such that* $\mathbb{D}_0 = \{-\infty_0, -\infty'_0\}$, $-\infty_0 \, \mathbb{D} \, -\infty'_0$ *and* $-\infty'_0, -\infty_0 < -\infty < ... < -1 < 0 < 1$. $A = (A, \vee, \wedge, \odot, \to, 1)$ *is an infinitely branchwise residuated skew lattice with the following operations:*
$-\infty_0 \wedge -\infty'_0 = -\infty_0$, $-\infty'_0 \wedge -\infty_0 = -\infty'_0$, $-\infty_0 \vee -\infty'_0 = -\infty'_0, -\infty'_0 \vee -\infty_0 = -\infty_0$. *Also for all* $x, y \in A$, *if* $x \leq y$, *then* $x \wedge y = x$ *and* $x \vee y = y$.

$\to$	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	0	1
$-\infty_0$	1	1	1	$\ldots$	1	1	1	1	1
$-\infty'_0$	1	1	1	$\ldots$	1	1	1	1	1
$-\infty$	$-\infty_0$	$-\infty'_0$	1	$\cdots$	1	1	1	1	1
$\vdots$	$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$		
-3	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	1	1	1	1	1
-2	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	1	1	1	1
-1	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	1	1	1
0	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	1	1
1	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	0	1

$\odot$	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	0	1
$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$\ldots$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$
$-\infty'_0$	$-\infty_0$	$-\infty'_0$	$-\infty'_0$	$\ldots$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$
$-\infty$	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$		
-3	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-3	-3	-3	-3
-2	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-2	-2	-2
-1	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	-1	-1
0	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	0	0
1	$-\infty_0$	$-\infty'_0$	$-\infty$	$\ldots$	-3	-2	-1	0	1



Above example is a conormal skew chain that is distributive and any upset $(\uparrow u)$ is a distributive residuated lattice and satisfies in conditions $(4), (5)$ of Definition 5.1.

Lemma 5.2. *If $(A, \vee, \wedge, 1)$ is a conormal skew lattice with top 1 that $x \to y = (y \vee x \vee y) \to_y y$ and $(A, \odot, 1)$ is a commutative monoid, then A is a conormal residuated skew lattice that $x \to y = (y \vee x \vee y) \to_y y$ iff*
(1) $x \wedge (y \to x \odot y) \wedge x = x$,
(2) $y \vee ((x \to y) \odot x) \vee y = y$,
(3) $x \to (y \vee x \vee y) = 1$,
(4) $(z \to x) \to (z \to y \vee x \vee y) = 1$,
(5) $((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) \vee ((x \wedge y \wedge x) \odot z)) \vee ((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) = (x \odot z) \wedge (y \odot z) \wedge (x \odot z)$.

Proof. Let A be a conormal residuated skew lattice that $x \to y = (y \vee x \vee y) \to_y y$. Then
(1) By Definition 3.1 (3), $x \preceq (y \to (x \odot y))$ therefore $x \wedge (y \to (x \odot y)) \wedge x = x$.
(2) Since $x \to y \preceq x \to y$, then according to Definition 3.1 (3), $x \odot (x \to y) \preceq y$ therefore $y \vee (x \odot (x \to y)) \vee y = y$.
(3) Since $x \preceq y \vee x \vee y$ then by Lemma 3.1, $x \to y \vee x \vee y = 1$.
(4) Since $x \preceq y \vee x \vee y$ implies $z \to x \preceq z \to y \vee x \vee y$ implies $(z \to x) \to (z \to y \vee x \vee y) = 1$ (by Lemma 3.1).

(5) By Theorem 3.4, $(x \wedge y \wedge x) \odot z \preceq (x \odot z \wedge y \odot z \wedge x \odot z)$ it implies

$((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) \vee ((x \wedge y \wedge x) \odot z) \vee ((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) =$

$((x \odot z) \wedge (y \odot z) \wedge (x \odot z))$.
Conversely, replace in the definition of a conormal residuated skew lattice the adjointness condition by $(6-8)$.
(6) $x \preceq y$ implies $x \odot z \preceq y \odot z$. If $x \preceq y$, then $x \wedge y \wedge x = x$. (5) implies

$$\begin{aligned} ((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) = \\ ((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) \vee (x \odot z) \vee ((x \odot z) \wedge (y \odot z) \wedge (x \odot z)) = \\ x \odot z. \end{aligned}$$

that this implies $x \odot z \preceq y \odot z$.
(7) $x \preceq y$ iff $x \to y = 1$. If $x \preceq y$, then $y \vee x \vee y = y$. According to (3), $1 = x \to y \vee x \vee y = x \to y$. If $x \to y = 1$, then by (2), $y \vee (1 \odot x) \vee y = y$. It implies $x \preceq y$.
(8) If $x \preceq y$, then $z \to x \preceq z \to y$. It is clear according to $x \preceq y$ iff $y \vee x \vee y = y$ and $(4), (7)$.
(9) $x \odot z \preceq y$ iff $z \preceq x \to y$. If $z \preceq x \to y$, then by $(6), (2)$, $x \odot z \preceq x \odot (x \to y) \preceq y$. Which implies $x \odot z \preceq y$ so $x \to x \odot z \preceq x \to y$ by (8). It implies $z \preceq x \to y$ by (1). □

Theorem 5.1. *The class of all conormal residuated skew lattices that $x \to y = (y \vee x \vee y) \to_y y$ forms a variety.*

Proof. In previous lemma we showed that equalities $(1-5)$ are equivalent to $(x \odot z \preceq y$ iff $z \preceq x \to y)$. On the other hand $(A, \vee, \wedge, 1)$ is a conormal skew lattice with top 1 and $(A, \odot, 1)$ is a commutative monoid i.e. $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $x \wedge x = x$, $x \vee x = x$, $x \wedge (x \vee y) = x = x \vee (x \wedge y)$, $(x \wedge y) \vee y = y = (x \vee y) \wedge y$, and $x \odot 1 = 1 \odot x = x$. Therefore A is equational. □

Remark 5.1. *If A is a branchwise residuated skew lattice and $u, v \in A$, then $(\uparrow u) \cup (\uparrow v)$ is not a distributive residuated lattice but $(\uparrow u) \cap (\uparrow v)$ is a distributive residuated lattice.*

In Example 3.1, A is a residuated skew lattice which is not conormal, since $a = a \vee b \vee a \vee m \neq a \vee a \vee b \vee m = b$. Therefore by Proposition 2.2, there is a $(\uparrow u)$ such that is not lattice so $(\uparrow u)$ is not distributive residuated lattice. Therefore A is not branchwise residuated skew lattice.

Remark 5.2. *Any branchwise residuated skew lattice is conormal.*

Proposition 5.1. *Relation* $\mathbb{D}$ *(*$x \,\mathbb{D}\, y$ *iff* $x \preceq y$, $y \preceq x$*) is a congruence relation on branchwise residuated skew lattice.*

Proof. Let A be a branchwise residuated skew lattice. Since $\mathbb{D}$ is a congruence for distributive skew lattices with a top element, we only need to prove if $x_1 \,\mathbb{D}\, y_1$ and $x_2 \,\mathbb{D}\, y_2$, then $(x_1 \to x_2) \,\mathbb{D}\, (y_1 \to y_2)$ and $(x_1 \odot x_2) \,\mathbb{D}\, (y_1 \odot y_2)$ holds for every $x_1, x_2, y_1, y_2 \in A$. Without loss of generality, we may assume $x_2 \leq x_1$ and $y_2 \leq y_1$. (Otherwise replace x_1 by $x_2 \vee x_1 \vee x_2$ and y_1 by $y_2 \vee y_1 \vee y_2$.) We define a map $\varphi : \uparrow x_2 \longrightarrow \uparrow y_2$ by setting $\varphi(x) = y_2 \vee x \vee y_2$. We claim that φ is a lattice isomorphism of $(\uparrow x_2, \vee, \wedge)$ with $(\uparrow y_2, \vee, \wedge)$, with inverse $\psi : \uparrow y_2 \longrightarrow \uparrow x_2$ given by $\psi(y) = x_2 \vee y \vee x_2$. It is easily seen that φ and ψ are inverses of each other. For instance,

$$\psi(\varphi(x)) = x_2 \vee y_2 \vee x \vee y_2 \vee x_2 = (x_2 \vee y_2 \vee x_2) \vee x \vee (x_2 \vee y_2 \vee x_2) = x_2 \vee x \vee x_2 = x$$

(since any skew lattice is a regular band, $x_2 \,\mathbb{D}\, y_2$ and $x \in (\uparrow x_2)$). φ must preserve $\vee, \wedge$. Indeed by distributive condition:

$$\varphi(x \wedge x') = y_2 \vee (x \wedge x') \vee y_2 = (y_2 \vee x \vee y_2) \wedge (y_2 \vee x' \vee y_2) = \varphi(x) \wedge \varphi(x').$$

Since any skew lattice is a regular band, then

$$\begin{aligned}
\varphi(x \vee x') &= y_2 \vee (x \vee x') \vee y_2 \\
&= y_2 \vee x \vee y_2 \vee x' \vee y_2 \\
&= y_2 \vee x \vee y_2 \vee y_2 \vee x' \vee y_2 \\
&= (y_2 \vee x \vee y_2) \vee (y_2 \vee x' \vee y_2) \\
&= \varphi(x) \vee \varphi(x').
\end{aligned}$$

Thus φ (and ψ) is a lattice isomorphism of $(\uparrow x_2)$ with $(\uparrow y_2)$. But then φ and ψ are also isomorphisms of residuated lattices. That is $\varphi(x \to y) = \varphi(x) \to \varphi(y)$ and $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$. Next, observe that $x \,\mathbb{D}\, \varphi(x)$, for all $x \in (\uparrow x_2)$. Indeed, any skew lattice is a regular band gives:

$$\varphi(x) \vee x \vee \varphi(x) = (y_2 \vee x \vee y_2) \vee x \vee (y_2 \vee x \vee y_2) = y_2 \vee x \vee y_2 = \varphi(x)$$

and likewise

$$x \vee \varphi(x) \vee x = \psi(\varphi(x)) \vee \varphi(x) \vee \psi(\varphi(x)) = \psi(\varphi(x)) = x.$$

x_1 is the unique element in its $\mathbb{D}$-class belonging to $(\uparrow x_2)$ and y_1 is the unique element in the same $\mathbb{D}$-class belonging to $(\uparrow y_2)$ (since each upset $(\uparrow u)$

intersects any $\mathbb{D}$-class in at most one element). But $\varphi(x_1)$ in $(\uparrow y_2)$ behaves in the manner just like y_1, and so $\varphi(x_1) = y_1$. Since $x_2 \mathbb{D} y_2$, $\varphi(x_2) = y_2 \vee x_2 \vee y_2 = y_2$ and $\varphi(x_1 \to x_2) = \varphi(x_1) \to \varphi(x_2) = y_1 \to y_2$, thus giving $x_1 \to x_2 \mathbb{D} y_1 \to y_2$. $\varphi(x_1 \odot x_2) = \varphi(x_1) \odot \varphi(x_2) = y_1 \odot y_2$, therefore $x_1 \odot x_2 \mathbb{D} y_1 \odot y_2$. □

Proposition 5.2. *Let A be a distributive skew lattice with top element 1. If $(A, \vee, \wedge, \odot, \to, 1)$ is a branchwise residuated skew lattice with $\odot, \to$, then $(A/\mathbb{D}, \vee, \wedge, \odot, \to, 1_{A/\mathbb{D}})$ is a generalized distributive residuated lattice with the same $\odot, \to$. ($A/\mathbb{D}$ is maximal lattice image of A).*

Proof. The induced homomorphism $\varphi : A \to A/\mathbb{D}$ is bijective on any commutative subset of A since distinct commuting elements of A lie in distinct $\mathbb{D}$-classes. It follows that for every $u \in A$, φ restricts to an isomorphism of upsets, $\varphi_u : \uparrow u \;\cong \uparrow \varphi(u)$. Thus each upset $(\uparrow u)$ in A forms a distributive residuated lattice if and only if each upset $(\uparrow v)$ in $A/\mathbb{D}$, being some $(\uparrow \varphi(u))$, must form a distributive residuated lattice. Since A is a distributive skew lattice, then $A/\mathbb{D}$ is a distributive lattice, because any distributive skew lattice is a quasi-distributive. Therefore $A/\mathbb{D}$ is a distributive lattice that each upset $(\uparrow v)$ is a distributive residuated lattice. Hence $(A/\mathbb{D}, \vee, \wedge, \odot, \to, 1_{A/\mathbb{D}})$ is a generalized distributive residuated lattice. □

Theorem 5.2. *If A is a conormal distributive residuated skew lattice which $x \odot y \mathbb{D} x \odot_u y$, for every $u \in A$, $x, y \in (\uparrow u)$, then Definitions 3.1 and 5.1 are equivalent i.e. branchwise residuated skew lattice and residuated skew lattice are equivalent.*

Proof. Suppose that A is a branchwise residuated skew lattice. It is enough to show that $x \odot y \preceq z$ iff $x \preceq y \to z$. Since the induced epimorphism $\varphi : A \to A/\mathbb{D}$ is a homomorphism of branchwise residuated skew lattices, we have
$x \preceq y \to z$ iff $\varphi(x) \leq \varphi(y) \to \varphi(z)$ iff $\varphi(x) \odot \varphi(y) \leq \varphi(z)$ iff $x \odot y \preceq z$.
Let A be a residuated skew lattice. Suppose that x, y, z lie in a common $(\uparrow u)$. Since $\preceq$ coincide to $\leq$ in $(\uparrow u)$ and $y \to z$ lies in $(\uparrow u)$ by Lemma 3.1 we have $x \leq y \to z$ iff $x \odot y \leq z$ in $(\uparrow u)$. Then $(\uparrow u, \wedge, \vee, \to, \odot, u, 1)$ is a residuated lattice. Since A is distributive, then $(\uparrow u)$ is distributive too. Now, consider the derived implication $\to^*$ given by $x \to^* y = (y \vee x \vee y) \to_y y$. By assumption both $y \to z$ and $y \to^* z$ satisfy $x \odot y \preceq z$ iff $x \preceq y \to z$ and thus are $\mathbb{D}$-equivalent. But since both lie in the sublattice $\uparrow z$ and A is conormal, they must be equal. □

Since any branchwise residuated skew lattice is a residuated skew lattice, then all of the properties which were stated in residuated skew lattices will hold in branchwise residuated skew lattices.

6 CONCLUSION

Skew Boolean algebras and normal skew lattices were defined by Leech and skew Heyting algebras were defined by Cvetko-Vah. In this paper, the residuum condition was applied to skew lattice and residuated skew lattice was defined as an extension of residuated lattice. Its properties was investigated and it was shown that the class of all conormal residuated skew lattices that $x \to y = (y \vee x \vee y) \to_y y$ forms a variety. It was shown that Green's relation $\mathbb{D}$ is a congruence on residuated skew lattice A, and $A/\mathbb{D}$ is a residuated lattice. Deductive system and skew deductive system was defined in residuated skew lattice. Branchwise residuated skew lattice was defined and it was shown that Green's relation $\mathbb{D}$ is a congruence on it. It was shown that a conormal distributive residuated skew lattice is equivalent with a branchwise residuated skew lattice under a condition and maximal lattice image of a branchwise residuated skew lattice is a generalized distributive residuated lattice too.

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